4.1 USING FIRST AND SECOND DERIVATIVES

What Derivatives Tell Us About a Function and its Graph

As we saw in Chapter 2, the connection between derivatives of a function and the function itself is given by the following:

- If $f' > 0$ on an interval, then f is increasing on that interval. Thus, f is monotonic over that interval.
- If $f' < 0$ on an interval, then f is decreasing on that interval. Thus, f is monotonic over that interval.
- If $f'' > 0$ on an interval, then the graph of f is concave up on that interval.
- If $f'' < 0$ on an interval, then the graph of f is concave down on that interval.

We can do more with these principles now than we could in Chapter 2 because we now have formulas for the derivatives of the elementary functions.

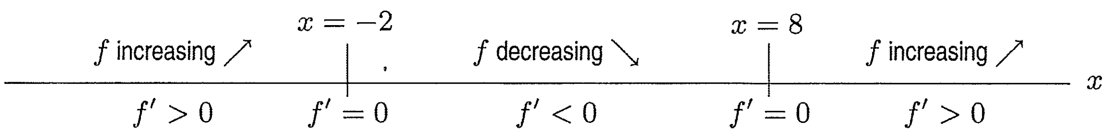
When we graph a function on a computer or calculator, we often see only part of the picture, and we may miss some significant features. Information given by the first and second derivatives can help identify regions with interesting behavior.

Example 1 Consider the function $f(x) = x^3 - 9x^2 - 48x + 52$.

Solution Since f is a cubic polynomial, we expect a graph that is roughly S-shaped. We can use the derivative to determine where the function is increasing and where it is decreasing. The derivative of f is

$$f'(x) = 3x^2 - 18x - 48.$$

To find where $f' > 0$ or $f' < 0$, we first find where $f' = 0$, that is, where $3x^2 - 18x - 48 = 0$. Factoring, we get $3(x + 2)(x - 8) = 0$, so $x = -2$ or $x = 8$. Since $f' = 0$ *only* at $x = -2$ and $x = 8$, and since f' is continuous, f' cannot change sign on any of the three intervals $x < -2$, or $-2 < x < 8$, or $8 < x$. How can we tell the sign of f' on each of these intervals? One way is to pick a point and substitute into f'. For example, since $f'(-3) = 33 > 0$, we know f' is positive for $x < -2$, so f is increasing for $x < -2$. Similarly, since $f'(0) = -48$ and $f'(10) = 72$, we know that f decreases between $x = -2$ and $x = 8$ and increases for $x > 8$. Summarizing:



We find that $f(-2) = 104$ and $f(8) = -396$. Hence on the interval $-2 < x < 8$ the function decreases from a high of 104 to a low of -396. One more point on the graph is easy to get: the y intercept, $f(0) = 52$. With just these three points we can get a graph. On a calculator, a window of $-10 \le x \le 20$ and $-400 \le y \le 400$ gives the graph in Figure 4.1.

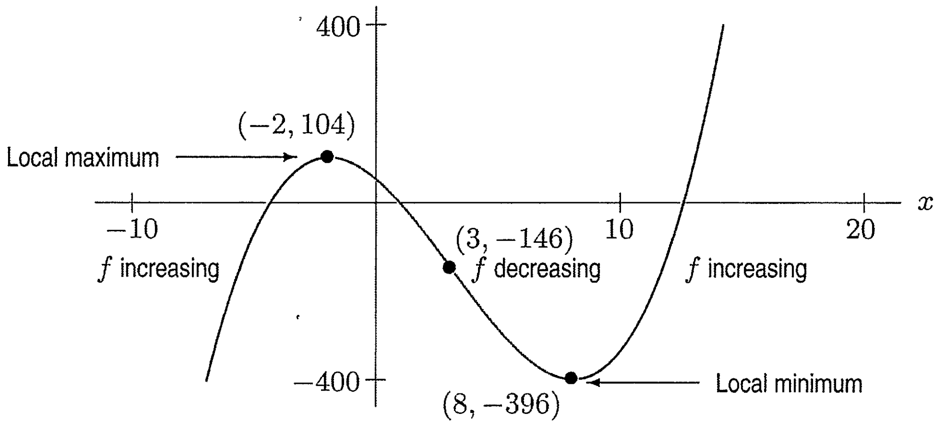


Figure 4.1: Useful graph of $f(x) = x^3 - 9x^2 - 48x + 52$

In Figure 4.1, we see that part of the graph is concave up and part is concave down. We can use the second derivative to analyze concavity. We have

$$f''(x) = 6x - 18.$$

Thus, $f''(x) < 0$ when $x < 3$ and $f''(x) > 0$ when $x > 3$, so the graph of f is concave down for $x < 3$ and concave up for $x > 3$. At $x = 3$, we have $f''(x) = 0$. Summarizing:

	$x = 3$	
f concave down $\cap$	$\vert$	f concave up $\cup$
$f'' < 0$	$f'' = 0$	$f'' > 0$

Local Maxima and Minima

We are often interested in points such as those marked local maximum and local minimum in Figure 4.1. We have the following definition:

> Suppose p is a point in the domain of f:
>
> - f has a **local minimum** at p if $f(p)$ is less than or equal to the values of f for points near p.
> - f has a **local maximum** at p if $f(p)$ is greater than or equal to the values of f for points near p.

We use the adjective "local" because we are describing only what happens near p. Local maxima and minima are sometimes called *local extrema.*

How Do We Detect a Local Maximum or Minimum?

In the preceding example, the points $x = -2$ and $x = 8$, where $f'(x) = 0$, played a key role in leading us to local maxima and minima. We give a name to such points:

> For any function f, a point p in the domain of f where $f'(p) = 0$ or $f'(p)$ is undefined is called a **critical point** of the function. In addition, the point $(p, f(p))$ on the graph of f is also called a critical point. A **critical value** of f is the value, $f(p)$, at a critical point, p.

Notice that "critical point of f" can refer either to points in the domain of f or to points on the graph of f. You will know which meaning is intended from the context.

Geometrically, at a critical point where $f'(p) = 0$, the line tangent to the graph of f at p is horizontal. At a critical point where $f'(p)$ is undefined, there is no horizontal tangent to the graph—there's either a vertical tangent or no tangent at all. (For example, $x = 0$ is a critical point for the absolute value function $f(x) = |x|$.) However, most of the functions we work with are differentiable everywhere, and therefore most of our critical points are of the $f'(p) = 0$ variety.

The critical points divide the domain of f into intervals on which the sign of the derivative remains the same, either positive or negative. Therefore, if f is defined on the interval between two successive critical points, its graph cannot change direction on that interval; it is either increasing or decreasing. The following result, which is proved on page 181, tells us that all local maxima and minima which are not at endpoints occur at critical points.

> ### Theorem 4.1: Local Extrema and Critical Points
>
> Suppose f is defined on an interval and has a local maximum or minimum at the point $x = a$, which is not an endpoint of the interval. If f is differentiable at $x = a$, then $f'(a) = 0$. Thus, a is a critical point.

Warning! The second-derivative test does not tell us anything if both $f'(p) = 0$ and $f''(p) = 0$. For example, if $f(x) = x^3$ and $g(x) = x^4$, both $f'(0) = g'(0) = 0$ and $f''(0) = g''(0) = 0$. The point $x = 0$ is a minimum for g but is neither a maximum nor a minimum for f. However, the first-derivative test is still useful. For example, g' changes sign from negative to positive at $x = 0$, so we know g has a local minimum there.

Concavity and Inflection Points

Investigating points where the slope changes sign led us to critical points. Now we look at points where the concavity changes.

A point at which the graph of a function changes concavity is called an **inflection point** of f. Suppose f' is continuous.

- A point p where f'' is zero or undefined is a possible inflection point.
- To test whether p is an inflection point, check whether f'' changes sign at p.

The words "inflection point of f" can refer either to a point in the domain of f or to a point on the graph of f. The context of the problem will tell you which is meant.

Example 5 Find the critical and inflection points for $g(x) = xe^{-x}$ and sketch the graph of g for $x \geq 0$.

Solution Taking derivatives and simplifying, we have

$$g'(x) = (1 - x)e^{-x} \quad \text{and} \quad g''(x) = (x - 2)e^{-x}.$$

So $x = 1$ is a critical point, and $g' > 0$ for $x < 1$ and $g' < 0$ for $x > 1$. Hence g increases to a local maximum at $x = 1$ and then decreases. Also, $g(x) \to 0$ as $x \to \infty$. There is an inflection point at $x = 2$ since $g'' < 0$ for $x < 2$ and $g'' > 0$ for $x > 2$. The graph is sketched in Figure 4.7.

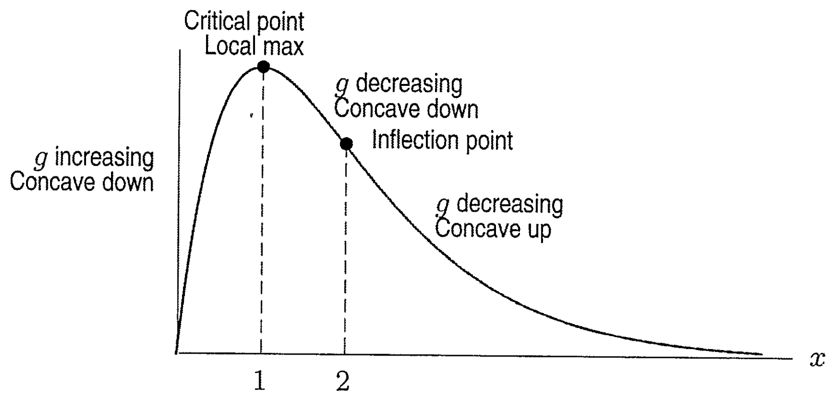


Figure 4.7: Graph of $g(x) = xe^{-x}$

Warning! Not every point x where $f''(x) = 0$ (or f'' is undefined) is an inflection point (just as not every point where $f' = 0$ is a local maximum or minimum). For instance $f(x) = x^4$ has $f''(x) = 12x^2$ so $f''(0) = 0$, but $f'' > 0$ when $x > 0$ and when $x < 0$, so there is *no* change in concavity at $x = 0$. See Figure 4.8.

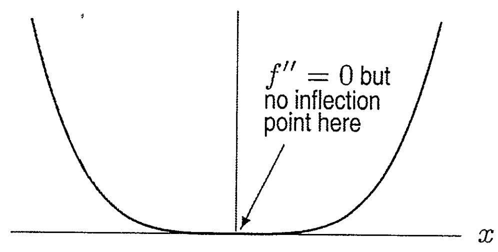


Figure 4.8: Graph of $f(x) = x^4$

Inflection Points and Local Maxima and Minima of the Derivative

Inflection points can also be interpreted in terms of first derivatives. Applying the First Derivative Test for local maxima and minima to f', we obtain the following result:

> Suppose a function f has a continuous derivative. If f'' changes sign at p, then f has an inflection point at p, and f' has a local minimum or a local maximum at p.

Figure 4.9 shows two inflection points. Notice that the curve crosses the tangent line at these points and that the slope of the curve is a maximum or a minimum there.

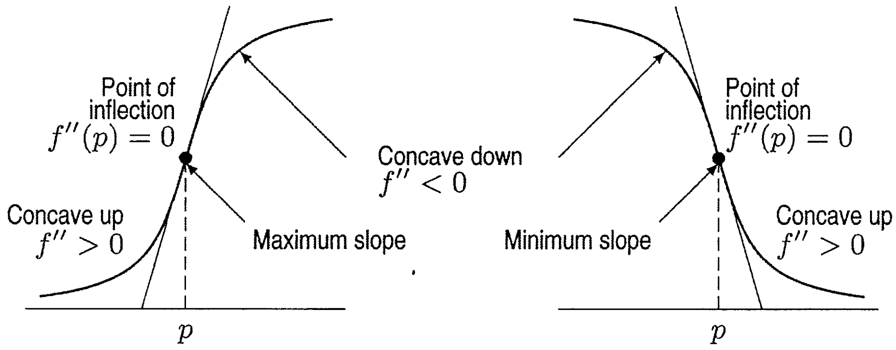


Figure 4.9: Change in concavity at p: Points of inflection

Example 6 Water is being poured into the vase in Figure 4.10 at a constant rate, measured in volume per unit time. Graph $y = f(t)$, the depth of the water against time, t. Explain the concavity and indicate the inflection points.

Solution At first the water level, y, rises slowly because the base of the vase is wide, and it takes a lot of water to make the depth increase. However, as the vase narrows, the rate at which the water is rising increases. Thus, y is increasing at an increasing rate and the graph is concave up. The rate of increase in the water level is at a maximum when the water reaches the middle of the vase, where the diameter is smallest; this is an inflection point. After that, the rate at which y increases decreases again, so the graph is concave down. (See Figure 4.11.)

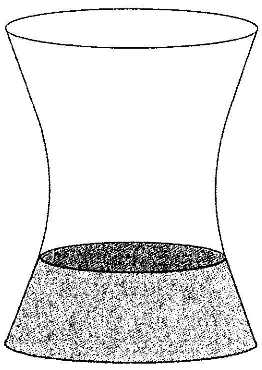


Figure 4.10: A vase

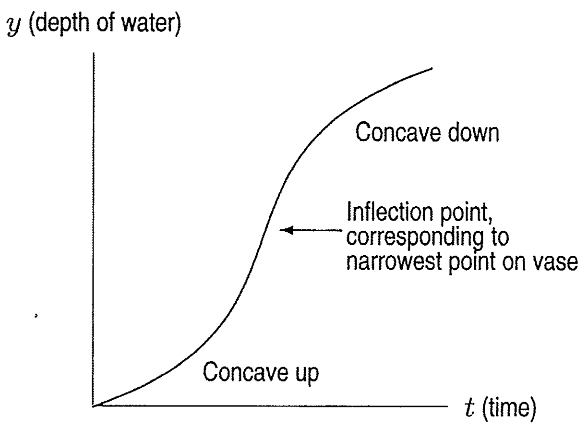


Figure 4.11: Graph of depth of water in the vase, y, against time, t

Showing Local Extrema Are At Critical Points

We now prove Theorem 4.1, which says that inside an interval, local maxima and minima can only occur at critical points. Suppose that f has a local maximum at $x = a$. Assuming that $f'(a)$ is defined, the definition of the derivative gives

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$